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# ALGEBRAICALLY CLOSED THEORIES

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# Algebraically closed theories

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**Abstract :** *The present work takes place in the study of infinitary behaviours for CCS-like communicating processes. A problem in that area arises from the fact that most of the abstraction morphisms we are interested in don't commute with least fixed points. In order to offer an alternative to least fixed point semantics we present an axiomatization of the notion of fixed point calculus within the formalism of algebraic theories. Such a calculus fixes one solution for each equation resulting from the interpretation of a set of recursive definitions in a way consistent with the free interpretation of the equations. This leads us to the notion of algebraically closed theory which stands for an algebraic theory equipped with a fixed point calculus. The rational theories by ADJ appear to be a special case of algebraically closed theories when least solutions are always chosen.*

# Les théories algébriquement closes

**Résumé :** *Une des difficultés rencontrées dans l' étude des comportements infinitaires de processus de type CCS provient du fait que les morphismes d'abstraction auxquels on s'intéresse ne commutent généralement pas avec les plus petits points fixes. Afin de trouver une alternative à la sémantique par plus petits points fixes nous proposons une axiomatisation de la notion de calcul de points fixes dans le cadre des théories algébriques. Un tel calcul est un procédé qui choisit une solution pour toute équation résultant de l 'interprétation d'un système de définitions récursives de façon cohérente avec l'interprétation libre de ces équations. Cela nous conduit à la notion de théorie algébriquement close ce qui désigne une théorie algébrique munie d'un tel calcul de points fixes. Comme cas particulier de théories algébriquement closes on retrouve les théories rationnelles introduites par ADJ et correspondant au choix des plus petites solutions pour ces équations.*

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# Algebraically closed theories

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## Abstract

*The present work takes place in the study of infinitary behaviours for CCS-like communicating processes. A problem in that area arises from the fact that most of the abstraction morphisms we are interested in don't commute with least fixed points. In order to offer an alternative to least fixed point semantics we present an axiomatization of the notion of fixed point calculus within the formalism of algebraic theories. Such a calculus fixes one solution for each equation resulting from the interpretation of a set of recursive definitions in a way consistent with the free interpretation of the equations. This leads us to the notion of algebraically closed theory which stands for an algebraic theory equipped with a fixed point calculus. The rational theories by ADJ appear to be a special case of algebraically closed theories when least solutions are always chosen.*

## 1 Introduction

Motivation for the present work are to be found in the study of infinitary behaviours of processes in CCS-like languages [Mil80]. Such languages are essentially defined by their signature $\Sigma = \bigcup_{n\in\omega} \Sigma_n$ ($\Sigma_n$ stands for the n-ary oprator symbols). Terms built on that signature denotes finite processes, whereas infinite processes are defined recursively, using for example the classical combinator **rec** : for instance, if $\Sigma_0 = \{nil\}$ and $\Sigma_1 = \{a.- \ / \ a \in A\}$ where A is a set of actions and $\Sigma_2 = \{+, |\}$, then the rational expression **let rec** $(x = \alpha.nil + \beta.x)$ **in** $\gamma.x$ is a finite description of the tree obtained by unfolding infinitely many times the definition $x = \alpha.nil + \beta.x$, namely

$$\gamma.(\alpha.nil + \beta.(\alpha.nil + \beta(\alpha.nil + \beta \ldots)))$$

A model for such a language must provide, first of all, an interpretation of the operator symbols in a semantical domain D, supplying for each n-ary operator symbol $f \in \Sigma_n$ a corresponding n-ary operator on that domain $\delta_f : D^n \to D$, the $\Sigma$-algebra $\delta = \{\delta_f; f \in \Sigma\}$ procures an interpretation in D for every closed $\Sigma$ term. In the same way, a $\Sigma$-algebra defines implicitely for each system of recursive definitions, an associated system of equations on the domain D. For example, given the declaration **rec** $(x = \alpha.nil + \beta.x)$ there corresponds the equation $e : \xi = \delta_+(\delta_\alpha(\delta_{nil}), \delta_\beta(\xi))$ on D. For that reason, a model must supply a solution $e^\dagger$ for each equation $e$ resulting from the interpretation of a system of recursive definitions. Let the corresponding process be called a *fixed point calculus*. In such a model, the interpretation of the rational expression given above is now fully defined as $\delta_\gamma(e^\dagger)$. Then

$$\text{a model} = \text{ a } \Sigma\text{-algebra} + \text{a fixed point calculus}$$

In most denotational models, D is an $\omega$-complete ordered set (i.e. $u_0 \sqsubseteq u_1 \sqsubseteq \ldots \sqsubseteq u_n \sqsubseteq \ldots$ has a least upper bound $\coprod u_i$ in D) with a least element $\perp$, where operator symbols are interpreted by $\omega$-continuous operators ($f(\coprod u_i) = \coprod f(u_i)$ for every increasing chain $(u_i; i \in \omega)$). The Tarski's theorem then ensures the existence of least fixed points, computed as least upper bounds of inductive chains. In the example, $e^\dagger$ is the least upper bound of the chain

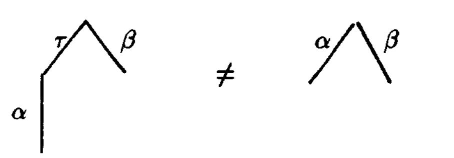
$$\perp \sqsubseteq e(\perp) \sqsubseteq e(e(\perp)) \sqsubseteq \ldots \sqsubseteq e^n(\perp) \sqsubseteq \ldots$$

The elements of the domain D are partial descriptions of objects, $\perp$ standing for the absence of information and the order $\sqsubseteq$ expressing the fact that a partial description is less accurate than another one.

As noticed in [ADJ76] the method that consists in computing fixed points of transformations by iterating them from the least element may work even for non $\omega$-continuous theories ; theories in which such iterations provide always fixed points of transformations are called rational theories. An equivalent formulation has been given by Tiuryn using *regular algebras* [Tiu77] ; this type of semantics is usually called **least fixed point semantics**.

Least fixed point semantics may be used for operational models of programs, where the objects of D are, for instance, synchronization trees and in which all the actions of the processes are represented. Now, following [DG87] we intend to derive therefrom a family of models through abstraction morphisms.

We may forget information because we consider that a certain kind of action is unobservable or that some distinctions that appear between processes at the operational level become unrelevant at a more abstract level. Unfortunately, as soon as we are concerned with infinite behaviours, it occurs that abstraction morphisms generally don't commute with least fixed points : least fixed point semantics are there no longer suitable. A simple example of that situation arises when we consider the invisible action $\tau$ of CCS which corresponds to an '*internal*' action of a process. At the observational level we cannot simply erase all occurrences of internal actions, because a process which is likely to do an internal action is *unstable* and unstability has observable effects. For example, we don't want to identify the processes with respective synchronization trees




$$\tau.\alpha.nil + \beta.nil \quad \neq \quad \alpha.nil + \beta.nil$$

since the former can perform an internal action and then become unable to do the action $\beta$. However, we don't want to distinguish two successive internal actions from one. Thus, we factor the operational model of synchronization trees by the least congruence $\approx$ for which

$$\tau.\tau.p \approx \tau.p$$

In that way, we distinguish between a finite sequence of internal actions (unstability) from an infinite sequence of internal actions (divergent process). Now the least (and actually unique) solution of the equation $x = \tau.x$ in the initial model is $\tau^\omega$ , whereas all the unstable processes (s.t. $p \approx \tau.p$) are solutions in the factor model, hence the least solution is the class of $\tau.\perp$ and not the one of $\tau^\omega$.

This example shows that if we are looking for a general framework allowing us to describe the behaviours of processes at different levels of abstraction we must be able to handle fixed points that are not the least ones. There lays our motivation for going beyond the rational theories. In order to extend the ADJ approach we propose axioms for fixed point calculi within the formalism of algebraic theories. For us, a model is then an *algebraically closed theory* (that is to say an algebraic theory supplied with a fixed point calculus) together with a $\Sigma$-algebra structure on that theory. As for model morphisms, they must on one hand respect the interpretation of the operators (i.e. be congruences for those operators) on the other hand they must commute with the respective fixed point calculi defined on those models.

A particular role is played here by the Herbrand model, whose domain consists of the infinite rational trees built on the signature $\Sigma$. That model gives a '*minimal*' interpretation of the language in that the operator symbols are freely interpreted. In the Herbrand model, each n-ary operator symbol $f \in \Sigma_n$ is sent to the mapping which builds the tree $f(t_0, \ldots t_{n-1})$ from the n-uple of rational trees $(t_0, \ldots, t_{n-1})$ ; and the fixed point calculus associates to declarations the vectors of rational trees obtained by unfolding them ad infinity. The initiality of the Herbrand model means that every other model must be obtained as a morphic image of it. Otherwise stated, if two expressions stand for the same rational tree, then they must have the same interpretation in all models.

This condition is the only constraint : a fixed point calculus is a calculus which supplies one solution for each equation resulting from the interpretation of a set of recursive definitions, consistently with the free interpretation of those equations. Hence, we are looking for **coherence conditions** that would be strong enough to ensure the initiality of the Herbrand model and weak enough to allow for any reasonable fixed point calculus. For example, we must retrieve rational theories as a particular case of algebraically closed theories.

## 2 Lawvere's algebraic theories

We shall work within the formalism of algebraic theories as they have been introduced by Lawvere in [Law63]. If D is a non empty set, which will serve as semantical domain, a typical example of algebraic theory is the cartesian category associated to that set. It is defined as follow

**Example 1** *The algebraic theory* **Cart***[D] associated to D is composed, for each pair (n,m) of integers, of a set* **Cart***[D](n,m) of* arrows *from n to m ; as it is, these are the mappings from $D^n$ to $D^m$. Those arrows may be composed : this is the usual composition of mappings. Moreover, we have corresponding to each integer n the cartesian projections $\pi_i^n : D^n \rightarrow D$ that select one component from an n-uple*

$$\pi_i^n(d_0, \ldots, d_{n-1}) = d_i$$

*Those projections provide a* **finite product structure** *which means that we have a bijective correspondance*

$$(D^p \longrightarrow D^n) \equiv (D^p \longrightarrow D)^n$$

*between the set of arrows from p to n and the set of n-ulpes of arrows from p to 1. Actually, from one hand we can split any arrow f from p to n into an n-uple of arrows $f_i$ from p to 1 obtained as the composites of f with the corresponding projections*

$$f_i = \pi_i^n \circ f : D^p \longrightarrow D^n \longrightarrow D$$

*on the other hand, we can juxtapose n mappings $f_i : D^p \rightarrow D$ component by component to obtain a mapping $[f_0, \ldots, f_{n-1}] : D^p \longrightarrow D^n$. Those two operations are mutually inverse which means that the arrow $[f_0, \ldots, f_{n-1}]$ is the unique mapping from $D^p$ to $D^n$ such that for each index i we have*

$$\pi_i^n \circ [f_0, \ldots, f_{n-1}] = f_i$$

More generally, Lawvere defined algebraic theories as those categories whose objects are the natural numbers and in which each object $n$ is the categorical product of the object 1 with itself $n$ times. For notational convenience we shall identify the natural numbers with the finite ordinals : $0 \equiv \emptyset$ and if $n > 0$ we have $n \equiv \{0, \ldots, n-1\}$.

**Definition 2** *A Lawvere's algebraic theory consists, for each pair (n,m) of natural numbers, of a set $\mathcal{T}(n,m)$ of arrows from n to m (we write $f : n \rightarrow m$ for $f \in \mathcal{T}(n,m)$) and a composition operation $\circ$ : $\mathcal{T}(m,p) \times \mathcal{T}(n,m) \rightarrow \mathcal{T}(n,p)$ such that :*

1. *it is a category, which means that :*
   - *(a) for each integer n, there exists an arrow $1_n : n \rightarrow n$ such that :*
     - *i. for every arrow $f : n \rightarrow p$ one has : $f \circ 1_n = f$*
     - *ii. for every arrow $g : m \rightarrow n$ one has : $1_n \circ g = g$*
   - *(b) associativity of composition : for every arrows $f : n \rightarrow m$, $g : m \rightarrow p$ and $h : p \rightarrow q$ one has $h \circ (g \circ f) = (h \circ g) \circ f$*
2. *for every integer n, there exists n arrows $\pi_i^n$ : $n \rightarrow 1$ for $i \in n$ such that if $(f_i, i \in n)$ is an n-uple of arrows from p to 1, there exists a unique arrow from p to n denoted $[f_0, \ldots, f_{n-1}]$ such that for every $i \in n$ one has*

$$\pi_i^n \circ [f_0, \ldots, f_{n-1}] = f_i$$

The arrows from n to 1 constitute the *n-ary operators* of the theory, for example the projections $\pi_i^n$ (for $i \in n$) are n-ary operators. Every arrow $f : p \rightarrow n$ can be decomposed in a unique way as $f = [f_0, \ldots, f_{n-1}]$ in which its $i^{th}$ component is given by $f_i = \pi_i^n \circ f$ ; so f appears as an n-uple of p-ary operators.

$$\mathcal{T}(p,n) \equiv \mathcal{T}(p,1)^n$$

In the particular case where $n = 0$ we deduce there exists a unique arrow from p to 0 ; we denote it $0_p : p \rightarrow 0$.

We observe, moreover, that the object $n + m$ is the categorical product (!) of the objects $n$ and $m$ ; that is to say

$$\mathcal{T}(p,n) \times \mathcal{T}(p,m) \equiv \mathcal{T}(p,n+m)$$

actually, we define the two projections

$$\begin{array}{rl} & \pi_1^{n,m} = [\pi_0^{n+m}, \ldots, \pi_{n-1}^{n+m}] \\ \text{and} & \pi_2^{n,m} = [\pi_n^{n+m}, \ldots, \pi_{n+m-1}^{n+m}] \end{array}$$

i.e. $\pi_1^{n,m}$ picks up the n first components of n+m and $\pi_2^{n,m}$ the m last ones, then for every pair of arrows $f : p \rightarrow n$ and $g : p \rightarrow m$ there exists a unique arrow $[f,g] : p \rightarrow n + m$, namely

$$[f,g] = [\pi_0^n \circ f, \ldots, \pi_{n-1}^n \circ f, \pi_0^m \circ g, \ldots, \pi_{m-1}^m \circ g],$$

of which $f$ and $g$ are the two components ; that is to say :

$$\pi_1^{n,m} \circ [f,g] = f \quad \text{and} \quad \pi_2^{n,m} \circ [f,g] = g$$

Finally, if $f : n \rightarrow p$ and $g : m \rightarrow q$ are two arrows we define their product $f \times g : n + m \rightarrow p + q$ as

$$f \times g = [f \circ \pi_1^{n,m}, g \circ \pi_2^{n,m}].$$

In particular, one may check that

$$\pi_1^{n,m} = 1_n \times 0_m \quad \text{and} \quad \pi_2^{n,m} = 0_n \times 1_m.$$

Let us see two other examples of algebraic theories :

**Example 3** *Let $\mathcal{D}$ be the algebraic theory whose arrows $\mathcal{D}(n,m)$ from n to m are the mappings from m to n and in which the composition is the inverse composition of mappings. The only arrows from n to 1 are the projections $\pi_i^n$ given by $\pi_i^n(0) = i$ ; and then the arrows $f : n \rightarrow m$ are the m-uples whose components are projections. More generally, for any algebraic theory the arrows f whose components are all projections constitute the* **distinguished morphisms** *of the theory. In that way $\mathcal{D}$ may be identified with the subcategory of any algebraic theory that corresponds to its distinguished morphisms.*

**Example 4** *If $\Sigma$ is a signature, let $T(\Sigma, n)$ stands for the set of terms built from that signature and from the set n of variables. Let $Law(\Sigma)$ be the algebraic theory whose arrows $Law(\Sigma)(n, m) = T(\Sigma, n)^m$ are the m-uples of terms with variables in n (i.e. mappings from m to $T(\Sigma, n)$). The composition of arrows in $Law(\Sigma)$ is a Kleisli's product of those mappings defined as follow : if $f : n \rightharpoonup m$ and $g : p \rightharpoonup n$ are two arrows in $Law(\Sigma)$ that is to say, if $f : m \rightarrow T(\Sigma, n)$ and $g : n \rightarrow T(\Sigma, p)$ are both mappings, $f \circ g : p \rightharpoonup m$ is the mapping $g * f : m \rightarrow T(\Sigma, p)$ whose value in $j \in m$ is given by $(g * f)(j) = f(j)[g(i)/i \,;\, i \in n]$ ; that means that the $j^{th}$ component of $f \circ g$ is the term $f(j) \in T(\Sigma, n)$ in which we replace each occurrence of a variable from n by the term in $T(\Sigma, p)$ which is the component of g corresponding to that variable. The projection $\pi_i^n : n \rightharpoonup 1 \equiv 1 \rightarrow T(\Sigma, n)$ corresponds to the variable $i \in n$ considered as an elementary term of $T(\Sigma, n)$.*

A morphism of algebraic theories $F : \mathcal{T} \longrightarrow \mathcal{T}'$ takes each arrow of type $n \rightharpoonup m$ in the former theory to an arrow of the same type in the latter theory while preserving the structure of both. More precisely

**Definition 5** *A morphism $F : \mathcal{T} \rightarrow \mathcal{T}'$ between two algebraic theories is a functor which is the identity on objects and which preserves the distinguished finite product structures on $\mathcal{T}$ and $\mathcal{T}'$ ; that means that it consists in a mapping $F_{n,m} : \mathcal{T}(n, m) \rightarrow \mathcal{T}'(n, m)$ for each pair of integers, which satisfies the following conditions :*

1. *for every integer n, one has : $F_{n,n}(1_n) = 1_n$,*
2. *for every arrows $f : n \rightharpoonup m$ and $g : m \rightharpoonup p$ in $\mathcal{T}$ $F_{n,p}(g \circ f) = F_{m,p}(g) \circ F_{n,m}(f)$ and*
3. *for every integer n and $i \in n$ one has : $F_{n,1}(\pi_i^n) = \pi_i^n$.*

Let a $\Sigma$-algebra structure on an algebraic theory $\mathcal{T}$ be defined as an n-ary operator $\delta_f \in \mathcal{T}(n, 1)$ for each operator symbol $f \in \Sigma_n$ of arity n.
We observe that such a structure is nothing else than a morphism $\delta : Law(\Sigma) \rightarrow \mathcal{T}$. Actually, an operator symbol $f \in \Sigma_n$ may be identified with the element $f[0, \ldots, n-1]$ of $\mathcal{T}$(n,1) and then it corresponds to it an n-ary operator $\delta_f : n \rightharpoonup 1$ in $\mathcal{T}$. Conversely, if $\delta$ is a structure of $\Sigma$-algebra on $\mathcal{T}$, that is to say $\delta$ takes every n-ary operator symbol $f \in \Sigma_n$ to an n-ary operator $\delta_f : n \rightharpoonup 1$ in $\mathcal{T}$. We then define $\delta(t) : n \rightharpoonup 1$ for $t \in Law(\Sigma)(n, 1) \cong T(\Sigma, n)$ by induction :

$$\begin{aligned} \delta(i) &= \pi_i^n \\ \delta(f[t_0, \ldots, t_{n-1}]) &= \delta_f \circ [\delta(t_0), \ldots, \delta(t_{n-1})] \end{aligned}$$

then $\delta(t) : n \rightharpoonup m$ for $t \in Law(\Sigma)(n, m) = m \rightarrow T(\Sigma, n)$ component by component :

$$\delta(t) = [\delta(t_0), \ldots, \delta(t_{n-1})]$$

In the particular case where the target theory is the cartesian theory associated to a domain D, we observe that a morphism of algebraic theories $\delta$ : $Law(\Sigma) \longrightarrow \mathbf{Cart}[D]$ amounts to a $\Sigma$-algebra structure $\delta$ on D, i.e. to the data for each symbol $f \in \Sigma_n$ of a mapping $\delta_f : D^n \longrightarrow D$. More generally,

**Definition 6** *If $\mathcal{T}$ is a algebraic theory and $D$ a non empty set, a $\mathcal{T}$-algebra $\delta$ of domain $D$ is a morphism of algebraic theories $\delta : \mathcal{T} \rightarrow \mathbf{Cart}[D]$.*

In this way, if $\beta : \mathcal{T}' \rightarrow \mathcal{T}$ is a morphism of algebraic theories to each $\mathcal{T}$-algebra (D, $\delta$) corresponds a $\mathcal{T}'$-algebra $(D, \delta \circ \beta)$.
A $\mathcal{T}$-algebra (D,$\delta$) interprets any arrow $f : n \rightharpoonup m$ in $\mathcal{T}$ as a mapping from $D^n$ to $D^m$ ; otherwise stated : $\delta$ takes every arrow $f : n \rightharpoonup m$ and n-uple $a \in D^n$ to the m-uple $b \in D^m$ which is the result of the **action of** $f$ **on** $a$ given by : $b = f \bullet_{n,m} a = \delta_{n,m}(f)(a)$

$$\bullet_{(n,m)} : \mathcal{T}(n, m) \times D^n \longrightarrow D^m$$

$$f \bullet_{(n,m)} (a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{m-1})$$

This leads us to the following alternative definition of $\mathcal{T}$-algebras :

**Definition 7 ([Lin69])** *If $\mathcal{T}$ is an algebraic theory, a $\mathcal{T}$-algebra is a pair $(A, \bullet)$ in which $A$ is a non-empty set and $\bullet$ takes each pair (n,m) of integers to a mapping $\bullet_{(n,m)} : \mathcal{T}(n, m) \times A^n \longrightarrow A^m$ such that :*

- $\forall i \in n \quad \pi_i^n \bullet_{(n,1)} (a_0, \ldots, a_{n-1}) = a_i$
- *for any arrows $f : n \rightharpoonup m$ and $g : m \rightharpoonup p$ in $\mathcal{T}$ and $a \in A^n$ one has :*

$$(g \circ f) \bullet_{(n,p)} a = g \bullet_{(m,p)} (f \bullet_{(n,m)} a)$$

# 3 A fixed point calculus

In order to be able to handle infinite behaviours, we enrich the syntax by introducing a new operator symbol **rec** whose meaning is the following : if $f$ is an arrow from $n$+$p$ to $n$ , **rec**($f$) is then a mapping from $p$ to $n$ and corresponds to the interpretation of $f$ as a sytem of recursive definitions for the variables in $n$ in terms of themselves and of the parameters of $p$. The syntax is defined in terms of rational expressions as follows.

**Definition 8** *The set of rational expressions corresponding to a signature $\Sigma$ is the least set such that :*

1. *every arrow $f : n \rightharpoonup m$ in $Law(\Sigma)$ is an* **elementary rational expression** *of type $n \rightharpoonup m$,*

2. *if $f$ is a rational expression of type $n \rightharpoonup m$ and $g$ a rational expression of type $n \rightharpoonup p$ then $(f;g)$ is a rational expression of type $n \rightharpoonup m+p$,*

3. *if $f$ is a rational expression of type $n \rightharpoonup m$ and $g$ a rational expression of type $p \rightharpoonup q$ then $(f$* **and** *$g)$ is a rational expression of type $n+p \rightharpoonup m+q$,*

4. *if $f$ is a rational expression of type $n \rightharpoonup m$ and $g$ a rational expression of type $m \rightharpoonup p$ then (***let** *$f$* **in** *$g$) is a rational expression of type $n \rightharpoonup p$,*

5. *if $f$ is a rational expression of type $n+p \rightharpoonup n$ then* **rec***$(f)$ is a rational expression of type $p \rightharpoonup n$.*

*we let $Rat(\Sigma)(n,m)$ be the set of rational expressions of type $n \rightharpoonup m$ ; a program is a rational expression of type $0 \rightharpoonup 1$.*

And a meaning function w.r.t. a $\Sigma$-algebra structure is a function that satisfies the set of equations given in the following definition.

**Definition 9** *Let $(D,\delta)$ a $\Sigma$-algebra, a* **meaning function** *(w.r.t. $\delta$) is a mapping $\mathcal{M}$ that takes any rational expression $f$ of type $n \rightharpoonup m$ to a mapping $\mathcal{M}_{n,m}(f) : D^n \to D^m$ such that :*

1. *if $f \in Law(\Sigma)(n,m)$ then, for $\rho \in D^n$ $\mathcal{M}_{n,m}(f)(\rho) = \delta_{n,m}(f)(\rho)$,*

2. *if $f \in Rat(\Sigma)(n,m)$ and $g \in Rat(\Sigma)(n,p)$ then, for $\rho \in D^n$*

$$\mathcal{M}_{n,m+p}(f;g)(\rho) = \mathcal{M}_{n,m}(f)(\rho) \times \mathcal{M}_{n,p}(g)(\rho)$$

   *(up to the isomorphism $D^{m+p} \cong D^m \times D^p$),*

3. *if $f \in Rat(\Sigma)(n,m)$ and $g \in Rat(\Sigma)(p,q)$ then*

$$\mathcal{M}_{n+p,m+q}(f \textbf{ and } g)(\rho_1 \times \rho_2) = \mathcal{M}_{n,m}(f)(\rho_1) \times \mathcal{M}_{p,q}(g)(\rho_2)$$

   *for $\rho_1 \in D^n$ and $\rho_2 \in D^p$,*

4. *if $f \in Rat(\Sigma)(n,m)$ and $g \in Rat(\Sigma)(m,p)$ then, for $\rho \in D^n$*

$$\mathcal{M}_{n,p}(\textbf{let f in g})(\rho) = \mathcal{M}_{m,p}(g)(\mathcal{M}_{n,m}(f)(\rho))$$

5. *if $f \in Rat(\Sigma)(n+p,n)$ then, for $\rho \in D^p$*

$$\mathcal{M}_{p,n}(\textbf{rec(f)})(\rho) = \mathcal{M}_{n+p,n}(f)(\mathcal{M}_{p,n}(\textbf{rec(f)})(\rho) \times \rho)$$

Interpreting in an algebraic theory an arrow $f : n+p \rightharpoonup n$ as a system of recursive definitions for the variables in $n$ in terms of themselves and of the parameters of $p$ amounts to associate to it an arrow $f^\dagger : p \rightharpoonup n$ whose interpretation must be the following :


if $b = (b_0, \ldots, b_{n-1}) \in D^n$ is the result of the action of $f^\dagger$ on a p-uple $a = (a_0, \ldots, a_{p-1}) \in D^p$ for some $\mathcal{T}$-algebra structure on D then b must, as well, be the result of the action of $f$ on the $(n+p)$-uple whose $n$ first components are the $b_i$ and the last $p$ components are the $a_j$. That is to say :

$$\begin{aligned} b \;=\; f^\dagger \bullet_{p,n} a &= f \bullet_{n+p,n} (b \times a) \\ &= f \bullet_{n+p,n} ((f^\dagger \bullet_{p,n} a) \times a) \\ &= (f \circ [f^\dagger, 1_p]) \bullet_{p,n} a \end{aligned}$$

To solve this equation at the '*syntactical level*' (regarding $\mathcal{T}$ ) i.e. independently of the $\mathcal{T}$-algebra structure, amounts to find an arrow $f^\dagger \in \mathcal{T}(p,n)$ such that :

$$f^\dagger = f \circ [f^\dagger, 1_p]$$

The iterative theories introduced by Elgot [Elg75] are algebraic theories in which such an equation admits a unique solution as soon as f corresponds to a '*well-guarded equation*' which means that none of its component is a projection.
Another approach advocated by the group ADJ (cf [ADJ76]) makes use of rational theories ; those are algebraic theories in which the equation

$$f : n+p \rightharpoonup n \qquad \xi = f \circ [\xi, 1_p]$$

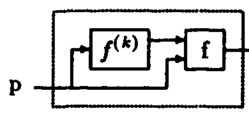
admits a '*least solution*' ; for that reason $\mathcal{T}$ must be an ordered theory i.e. the sets of arrows $\mathcal{T}(n,m)$ must be equipped with a partial order $\sqsubseteq_{n,m}$ with a least element $\perp_{n,m}$.

**Definition 10** *A* **rationally closed** *theory $\mathcal{T}$ is an ordered theory equipped with a mapping $\dagger : \mathcal{T}(n+p,n) \rightharpoonup \mathcal{T}(p,n)$ for each pair of integers $n$ and $m$. $f^\dagger$ is called the minimal solution for $f$ and satisfies the following conditions :*

- $f \circ [f^\dagger, 1_p] = f^\dagger$
- $f \circ [g, 1_p] \sqsubseteq_{p,n} g \Longrightarrow f^\dagger \sqsubseteq_{p,n} g$ *for every arrow $g : p \rightharpoonup n$*
- $(f \circ (1_n \times h))^\dagger = f^\dagger \circ h$ *for every arrow $h : m \rightharpoonup p$*

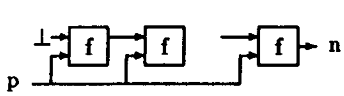
h "assigns a value" to the p parameters of the equation, the last condition states that it makes no difference if we solve an equation with parameters and then assigns values to those parameters (right-hand side of the equality) or else if we assigns values to the parameters before solving the equation (left-hand side of the equality).
The **iterates** of $f : n+p \rightharpoonup n$ being the arrows $f^{(k)} : p \rightharpoonup n$ given by :

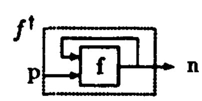
$$f^{(0)} = \bot_{p,n}$$
$$f^{(k+1)} = f \circ [f^{(k)}, 1_p]$$




a **rational theory** is then a rationally closed theory $\mathcal{T}$ for which $f^\dagger$ is the least upper bound (in $(\mathcal{T}(p,n); \sqsubseteq_{p,n})$) of the iterates of f

$$f^\dagger = \bigsqcup f^{(k)}$$




More generally we are looking for an axiomatic description of a fixed point calculus within an algebraic theory $\mathcal{T}$. It is a process that selects one solution $f^\dagger$ for each equation

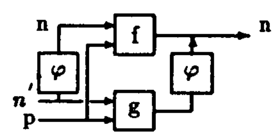
$$\xi = f \circ [\xi, 1_p]$$

associated to an arrow $f : n+p \rightharpoonup n$ in $\mathcal{T}$. When several solutions exist for such an equation the different choices taken by this process must be coherent in order to ensure consistency with the free interpretation of the language. This is to say that the same solution must be selected for two equations corresponding to rational expressions $f : n+p \rightharpoonup n$ and $g : n+p \rightharpoonup n$ that give rises by unfolding to the same n-uple of rational terms . In particular, the condition that appears in the case of rational theories and which states that the solution is *'free'* with respect to the parameters must be fulfilled. For the same reason it seems natural that this process would be compatible with the Gauss' method of resolution. That method consists in splitting a system of equations into two parts $f : n+m+p \rightharpoonup n$ and $g : n+m+p \rightharpoonup m$ ; in this way the unknowns of each system become additional parameters for the other one. We solve the former, it provides solutions for the variables in $n$ : $f^\dagger : m+p \rightharpoonup n$ in terms of the parameters of p and the variables m, by substitution in the latter we obtain the system $h = g \circ [f^\dagger, 1_{m+p}] : m+p \rightharpoonup m$ in which the variables n have disappeared. We readily verify that a solution for the initial system is given by $k = [f^\dagger \circ [h^\dagger, 1_p], h^\dagger]$. We want this solution to be selected by the fixed point calculus for the composite system, this is to say :

$$[f,g]^\dagger = [f^\dagger \circ [h^\dagger, 1_p], h^\dagger]$$

Which in particular shows that this process doesn't depend on the way we split the initial system. That condition is known as the *Scott-Bekič's condition* [Bek84]. Another constraint we require from a fixed point calculus concerns its compatibility with the reduction of a system of recursive definitions into an equivalent and simpler one. More precisely,

**Definition 11** *Let f and g be two systems of recursive definitions having respectively n and $n'$ as sets of variables and sharing the same set of parameters p ; that means that $f : n+p \rightharpoonup n$ and $g : n'+p \rightharpoonup n'$ are both arrows in $\mathcal{T}$. Let $\varphi : n' \rightharpoonup n$ be a distinguished morphism of $\mathcal{T}$ corresponding to a surjective mapping (from n onto $n'$). We say that g is a $\varphi$-reduction of f, which we denote as $f \xrightarrow{\varphi} g$ when the following commutes :*

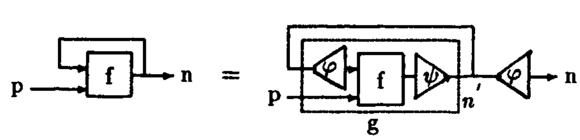



$$f \circ (\varphi \times 1_p) = \varphi \circ g$$

When $f \xrightarrow{\varphi} g$ we require that the solution selected for f is the $\varphi$ image of the solution selected for g. Since $\varphi$ is a distinguished morphism corresponding to a surjection from $n$ to $n'$, there exists some distinguished morphism $\psi : n \rightharpoonup n'$ such that $\psi \circ \varphi = 1_{n'}$ (we recall that composition of distinguished morphisms corresponds to the inverse composition of mappings). The constraint $\varphi \circ g = f \circ (\varphi \times 1_p)$ imposes $g = \psi \circ f \circ (\varphi \times 1_p)$, and then translates into

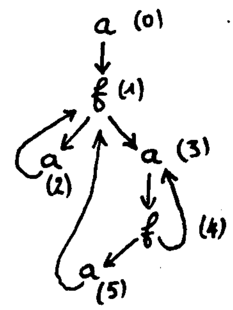
$$(*) \qquad (\varphi \circ \psi) \circ f \circ (\varphi \times 1_p) = f \circ (\varphi \times 1_p)$$

Compatibility with the reduction allows us to deduce the following identity :



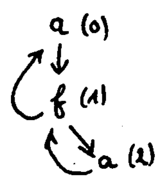
$\varphi$ and $\psi$ appear as *'coding'* and *'uncoding'* functions on the variables of the two systems and under the assumption (*) we may state that $f^\dagger$ and $g^\dagger$ have the same behaviour up to the *'changing of outputs'*. For example, let $f : 6 \rightharpoonup 6$ be an arrow in $Law(\Sigma)$ represented by the following system of equations (there is no parameters as $p = 0$)

$$
\begin{aligned}
x_0 &= a(x_1)\\
x_1 &= f(x_2, x_3)\\
x_2 &= a(x_1)\\
x_3 &= a(x_4)\\
x_4 &= f(x_5, x_3)\\
x_5 &= a(x_1)
\end{aligned}
$$




where $a \in \Sigma_1$ and $f \in \Sigma_2$.
$f^\dagger : 0 \rightharpoonup 6$ is the 6-uple whose components are obtained by unfolding the graph from the corresponding node. Now let $\varphi : 3 \rightharpoonup 6$ be defined by $\varphi(0) = \varphi(2) = \varphi(5) = 0$, $\varphi(1) = \varphi(4) = 1$ and $\varphi(3) = 2$ and let $\psi : 6 \rightharpoonup 3$ be one of its sections, for example : $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(2) = 3$ then $g : 3 \rightharpoonup 3 = \psi \circ f \circ (\varphi \times 1_0)$ corresponds to the following system of equations :

$$
\begin{aligned}
X_0 &= a(X_1)\\
X_1 &= f(X_0, X_2)\\
X_2 &= a(X_1)
\end{aligned}
$$




We readily verify that the condition (*) says that $\varphi$ is a *'congruence'* for $f$ and that $g$ is the corresponding quotient, otherwise stated $\varphi$ set up a *'bisimulation'* between the graphs associated to $f$ and $g$. And the last constraint we required for a fixed point calculus says that if a system $f$ can be reduced into a simpler system $g$, then the solution $f^\dagger$ selected for $f$ must be derived from the solution selected for $g$. In our example, both systems

$$
f : 6 \rightharpoonup 6 = \left\{ \begin{array}{lcl} x_0 &=& a(x_1)\\ x_1 &=& f(x_2, x_3)\\ x_2 &=& a(x_1)\\ x_3 &=& a(x_4)\\ x_4 &=& f(x_5, x_3)\\ x_5 &=& a(x_1) \end{array} \right.
$$

and $\varphi \circ g$ where

$$
\varphi : 3 \rightharpoonup 6 = \left\{ \begin{array}{lcccccl} x_0 &=& x_2 &=& x_5 &=& X_0\\ x_1 &=& x_4 &=& X_1 & & \\ x_3 &=& X_2 & & & & \end{array} \right.
$$

and

$$
g = \psi \circ (\varphi \times 1_0) = \left\{ \begin{array}{lcl} X_0 &=& a(X_1)\\ X_1 &=& f(X_0, X_2)\\ X_2 &=& a(X_1) \end{array} \right.
$$

must lead to the same solution.
Summing up the different constraints required for a fixed point calculus give us the following definition :

**Definition 12** *Let $\mathcal{T}$ be an algebraic theory, a* **fixed point calculus** *for $\mathcal{T}$ is a mapping $\dagger$ that takes any arrow $f : n + p \rightharpoonup n$ in $\mathcal{T}$ to an arrow $f^\dagger : p \rightharpoonup n$ in such a way that the following conditions are satisfied :*

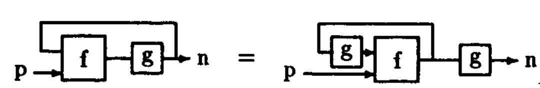
1. $f \circ [f^\dagger, 1_p] = f^\dagger$ *for any arrow* $f : n + p \rightharpoonup n$,
2. $(f \circ (1_n \times h))^\dagger = f^\dagger \circ h$
   *for any arrows* $f : n + p \rightharpoonup n$ *and* $h : m \rightharpoonup p$,
3. *- Scott-Bekič's condition -*
   *If* $f : n + m + p \rightharpoonup n$ *and* $g : n + m + p \rightharpoonup m$ *are both arrows in* $\mathcal{T}$ *then*
   $$[f, g]^\dagger = [f^\dagger \circ [h^\dagger, 1_p], h^\dagger] : p \rightharpoonup n + m$$
   $$\textit{where} \quad h = g \circ [f^\dagger, 1_{m+p}] : m + p \rightharpoonup m$$
4. *- Compatibility with the reductions -*
   *If* $f \xrightarrow{\varphi} g$ *then* $f^\dagger = \varphi \circ g^\dagger$

*An algebraically closed theory $(\mathcal{T}, \dagger)$ consists in an algebraic theory $\mathcal{T}$ together with a fixed point calculus $\dagger$ on $\mathcal{T}$.*

A fixed point calculus is compatible with the changing of variables

**Proposition 13** *If $f : n+p \rightharpoonup n'$ and $g : n' \rightharpoonup n$ are both arrows in an algebraically closed theory $(\mathcal{T}, \dagger)$ then*

$$(g \circ f)^\dagger = g \circ (f \circ (g \times 1_p))^\dagger$$




*Proof :*
Let $f : n + p \rightharpoonup n'$ and $g : n' \rightharpoonup n$ be two arrows in an algebraically closed theory $\mathcal{T}$, we set

$$
\begin{array}{lclcl}
F &=& f \circ (0_{n'} \times 1_n \times 1_p) &:& n' + n + p \rightharpoonup n'\\
G &=& g \circ (1_{n'} \times 0_n \times 0_p) &:& n' + n + p \rightharpoonup n
\end{array}
$$

We define the swap operation

$$\mathbf{swap}(n, m) = [\pi_2^{n,m}, \pi_1^{n,m}] : n + m \rightharpoonup m + n$$

swap(n,m) = 


and we let $\varphi = \mathbf{swap}(n, n')$, it has $\psi = \mathbf{swap}(n', n)$ as inverse arrow so if we let

$$\tilde{F} = F \circ (\varphi \times 1_p) = f \circ (1_n \times 0_{n'} \times 1_p)$$

$$\tilde{G} = G \circ (\varphi \times 1_p) = g \circ (0_n \times 1_{n'} \times 0_p)$$

we have $[\tilde{G}, \tilde{F}] = \psi \circ [F, G] \circ (\varphi \times 1_p)$ hence

$$[F, G]^\dagger = \varphi \circ [\tilde{G}, \tilde{F}]^\dagger$$

in particular

$$\pi_2^{n',n} \circ [F, G]^\dagger = \pi_1^{n,n'} \circ [\tilde{G}, \tilde{F}]^\dagger$$

Applying the Scott-Bekič condition to $[F, G]$ leads to

$$[F, G]^\dagger = [F^\dagger \circ [H^\dagger, 1_p], H^\dagger]$$

where $H = G \circ [F^\dagger, 1_{n+p}]$. We notice that the following identity holds, for $\alpha : m \rightarrow p$ and $f : n + p \rightarrow n$, in any algebraically closed theory

$$(f \circ (0_n \times \alpha))^\dagger = f \circ \alpha$$

actually

$$g = (f \circ (0_n \times \alpha))^\dagger = f \circ (0_n \times \alpha) \circ [g, 1_p] = f \circ \alpha$$

$$\text{hence} \quad \begin{array}{rcl} F^\dagger & = & f, \\ H & = & g \circ (1_{n'} \circ 0_n \times 0_p) \circ [f, 1_{n+p}] = g \circ f \end{array}$$

$$\pi_2^{n',n} \circ [F, G]^\dagger = (g \circ f)^\dagger$$

Now, applying the Scott-Bekič condition to $[\tilde{G}, \tilde{F}]$ leads to

$$[\tilde{G}, \tilde{F}]^\dagger = [\tilde{G}^\dagger \circ [K^\dagger, 1_p], K^\dagger]$$

where $K = \tilde{F} \circ [\tilde{G}^\dagger, 1_{n'+p}]$, it follows

$$\begin{array}{rcl} \tilde{G}^\dagger & = & g \circ (1_{n'} \times 0_p), \\ K & = & f \circ (1_n \times 0_{n'} \times 1_p) \circ [g \circ (1_{n'} \times 0_p), 1_{n'+p}] \\ & = & f \circ (g \times 1_p) \end{array}$$

and

$$\pi_1^{n,n'} \circ [\tilde{G}, \tilde{F}]^\dagger = g \circ (f \circ (g \times 1_p))^\dagger$$

hence

$$(g \circ f)^\dagger = g \circ (f \circ (g \times 1_p))^\dagger$$

□.

For example, if $g$ is a distinguished morphism corresponding to a bijection $(n = n')$ it shows that the order in which the equations are displayed has no influence on the computation of the solution. For a more involved example, let h be an arrow in $Law(\Sigma)$ that may be decomposed into $h = g \circ f$ where $f : 3 \rightarrow 4$, $g : 4 \rightarrow 3$ and $h : 3 \rightarrow 3$ correspond to the following systems of equations

$$h : 3 \rightarrow 3 = \left\{ \begin{array}{rcl} x_0 & = & \alpha(\beta(x_1), \alpha(x_2, \delta)) \\ x_1 & = & \gamma(x_2) \\ x_2 & = & \alpha(x_0, \gamma(x_2)) \end{array} \right.$$

$$g : 4 \rightarrow 3 = \left\{ \begin{array}{rcl} x_0 & = & \alpha(X_0, \alpha(X_1, \delta)) \\ x_1 & = & X_2 \\ x_2 & = & \alpha(X_3, X_2) \end{array} \right.$$

$$f : 3 \rightarrow 4 = \left\{ \begin{array}{rcl} X_0 & = & \beta(x_1) \\ X_1 & = & x_2 \\ X_2 & = & \gamma(x_2) \\ X_3 & = & x_0 \end{array} \right.$$

The solution selected for $h = g \circ f$ is the image by the changing of variables g of the solution selected for the system derived from f by this changing of variables, namely $f \circ (g \times 1_p)$.

$$f \circ (g \times 1_p) = \left\{ \begin{array}{rcl} X_0 & = & \beta(X_2) \\ X_1 & = & \alpha(X_3, X_2) \\ X_2 & = & \gamma(\alpha(X_3, X_2)) \\ X_3 & = & \alpha(X_0, \alpha(X_1, \delta)) \end{array} \right.$$

When $(\mathcal{T}, \dagger)$ is a rational theory, we verify that the mapping $\dagger$ that selects for each system its minimal solution is indeed a fixed point calculus according to our definition, that is to say :

**Proposition 14** *A rational theory is an algebraically closed theory.*

*Proof :*
The first two conditions are, by definition, fulfilled in any rational theories. The Scott-Bekič condition has been proved valid in any rational theories (cf [ADJ76]).

- Compatibility with the reductions -
Let $f : n + p \rightarrow n$ and $g : n' + p \rightarrow n'$ be two arrows in $\mathcal{T}$ and $\varphi : n \rightarrow n'$ a surjective mapping such that

$$f \circ (\varphi \times 1_p) = \varphi \circ g$$

Let $f^\dagger = \coprod f^{(k)}$ and $g^\dagger = \coprod g^{(k)}$ be the minimal solutions for $f$ and $g$ respectively. We prove by induction that

$$f^{(k)} = \varphi \circ g^{(k)}$$

- for $k = 0$ we have $\varphi \circ \perp_{p,n'} = \perp_{p,n}$ since $\varphi$ is a distinguished morphism,
- general case

$$\begin{array}{rcl} f^{(k+1)} & = & f \circ [f^{(k)}, 1_p] \\ & = & f \circ [\varphi \circ g^{(k)}, 1_p] \\ & = & f \circ (\varphi \times 1_p) \circ [g^{(k)}, 1_p] \\ & = & \varphi \circ g \circ [g^{(k)}, 1_p] \\ & = & \varphi \circ g^{(k+1)} \end{array}$$

hence $f^\dagger = \varphi \circ g^\dagger$ □.

# 4 A category of models

Roughly speaking, a model consists in a $\Sigma$-algebra structure that allows for the interpretation of the operator symbols together with a fixed point calculus that allows for the resolution of systems of recursive definitions

$$\text{a model} = \text{a } \Sigma\text{-algebra} + \text{a fixed point calculus}$$

As we have already noticed a $\Sigma$-algebra structure in an algebraic theory is none other that a morphism of algebraic theories $\alpha : Law(\Sigma) \to \mathcal{T}$ . And then :

**Definition 15** *A model is a triple $(\mathcal{T}, \dagger, \alpha)$ that corresponds to an algebraically closed theory together with a morphism of algebraic theories $\alpha : Law(\Sigma) \to \mathcal{T}$.*

And we may interpret a rational expression into a model.

**Definition 16** *If $(\mathcal{T}, \dagger, \alpha)$ is a model, for every pair of integer $(n,m)$ we define $\mathcal{M}^\alpha_{n,m} : Rat(\Sigma)(n,m) \to \mathcal{T}$ as follows :*

1. *if $f \in Law(\Sigma)(n,m)$ then $\mathcal{M}^\alpha_{n,m}(f) = \alpha(f)$,*
2. *if $f \in Rat(\Sigma)(n,m)$ and $g \in Rat(\Sigma)(n,p)$ then*
$$\mathcal{M}^\alpha_{n,m+p}(f;g) = [\mathcal{M}^\alpha_{n,m}(f), \mathcal{M}^\alpha_{n,p}(g)]$$
3. *if $f \in Rat(\Sigma)(n,m)$ and $g \in Rat(\Sigma)(p,q)$ then*
$$\mathcal{M}^\alpha_{n+p,m+q}(f \textbf{ and } g) = \mathcal{M}^\alpha_{n,m}(f) \times \mathcal{M}^\alpha_{p,q}(g)$$
4. *if $f \in Rat(\Sigma)(n,m)$ and $g \in Rat(\Sigma)(m,p)$ then*
$$\mathcal{M}^\alpha_{n,p}(\textbf{let } f \textbf{ in } g) = \mathcal{M}^\alpha_{m,p}(g) \circ \mathcal{M}^\alpha_{n,m}(f)$$
5. *if $f \in Rat(\Sigma)(n+p,n)$ then*
$$\mathcal{M}^\alpha_{p,n}(\textbf{rec}(f)) = (\mathcal{M}^\alpha_{n+p,n}(f))^\dagger$$

The following result may readily be established :

**Proposition 17** *If $\alpha : Law(\Sigma) \to \mathcal{T}$ is a morphism of algebraic theories, $\dagger$ a fixed point calculus on $\mathcal{T}$ and $(D,\delta)$ $\mathcal{T}$-algebra then*

$$\mathcal{M}^\alpha_{D,\delta} = Rat(\Sigma)(n,m) \xrightarrow{\mathcal{M}^\alpha_{n,m}} \mathcal{T}(n,m) \xrightarrow{\delta_{n,m}} (D^n \to D^m)$$

*is a meaning function w.r.t. the $\Sigma$-algebra structure $\delta \circ \alpha$ on $D$.*

The proof is by straightforward induction $\Box$.

In this way, for any $\mathcal{T}$-algebra $(D,\delta)$, the interpretation of the terms corresponding to the $\Sigma$-algebra $\delta \circ \alpha$ extends into an interpretation of the rational expressions.

$$\begin{tikzcd}
Rat(\Sigma)(n,m) \arrow[r, "\mathcal{M}^\alpha_{n,m}"] & \mathcal{T}(n,m) \arrow[dr, "\delta_{n,m}"] & \\
Law(\Sigma)(n,m) \arrow[u] \arrow[ur, "\alpha_{n,m}"] \arrow[rr, "(\delta \circ \alpha)_{n,m}"'] & & (D^n \to D^m)
\end{tikzcd}$$

A model morphism $\beta : (\mathcal{T}, \dagger, \alpha) \to (\mathcal{T}', \dagger', \alpha')$ is a morphism of algebraic theories from $\mathcal{T}$ to $\mathcal{T}'$ that, for one hand, respects the interpretation of the operators i.e. such that $\alpha' = \beta \circ \alpha$ and, for the other hand, commutes with the fixed point calculi which means :

**Definition 18** *We say that a morphism of algebraic theories $\beta : \mathcal{T} \to \mathcal{T}'$ commutes with the respective fixed point calculi $\dagger$ and $\dagger'$ on $\mathcal{T}$ and $\mathcal{T}'$ when the following diagram commutes :*

$$\begin{tikzcd}
\mathcal{T}(m+n,m) \arrow[r, "\dagger"] \arrow[d, "\beta_{m+n,m}"'] & \mathcal{T}(n,m) \arrow[d, "\beta_{n,m}"] \\
\mathcal{T}'(m+n,m) \arrow[r, "\dagger'"] & \mathcal{T}'(n,m)
\end{tikzcd}$$

Under that condition the following result can be established :

**Proposition 19** *Let $\alpha : Law(\Sigma) \to \mathcal{T}$ be a morphism of algebraic theories and $\beta : \mathcal{T} \to \mathcal{T}'$ a morphism of algebraic theories that commutes with the fixed point calculi on $\mathcal{T}$ and $\mathcal{T}'$ then*

$$\mathcal{M}^{\beta \circ \alpha} = \beta \circ \mathcal{M}^\alpha$$

*and then for any $\mathcal{T}$-algebra $(D,\delta)$ one has :*

$$\mathcal{M}^{\beta\circ\alpha}_{(D,\delta)} = \mathcal{M}^\alpha_{(D,\delta\circ\beta)}$$

$$\begin{tikzcd}
 & \mathcal{T}(n,m) \arrow[dd, "\beta_{n,m}"] \arrow[dr, "(\delta \circ \beta)_{n,m}"] & \\
Rat(\Sigma)(n,m) \arrow[ur, "\mathcal{M}^\alpha_{n,m}"] \arrow[dr, "\mathcal{M}^{\beta\circ\alpha}_{n,m}"'] & & (D^n \to D^m) \\
 & \mathcal{T}'(n,m) \arrow[ur, "\delta_{n,m}"'] &
\end{tikzcd}$$

That means that if $\beta : (\mathcal{T}, \dagger, \alpha) \to (\mathcal{T}', \dagger', \alpha')$ is a model morphism, the interpretation of a rational expression w.r.t. a $\mathcal{T}'$-algebra $\delta$ is the same as its interpretation relatively to the corresponding $\mathcal{T}$-algebra $\delta \circ \beta$ .
The proof is an easy verification $\Box$.

# 5 The free algebraically closed theories

We, first of all, define the algebraic theory $Law^*(\Sigma)$ freely generated by $\Sigma$ in a way reminiscent of Bloom and Elgot's construction of free iterative theories ([BE76]). The arrows of this algebraic theory are construe as equivalence classes of rational expressions

$$Law^*(\Sigma)(n,m) = Rat(\Sigma)(n,m)/\equiv$$

For this we define $\equiv$ as the least binary relation on the set of rational expressions satisfying the following conditions :

1. it is an equivalence relation, i.e. if f, g and h are arbitrary rational expressions one has :
   (a) $f \equiv f$,
   (b) $f \equiv g \Longrightarrow g \equiv f$,
   (c) $f \equiv g$ *et* $g \equiv h \Longrightarrow f \equiv h$;
2. rules that concern the substitution operation :
   (a) it is a congruence for the substitution operation :
   $$f \equiv f' \text{ and } g \equiv g' \Longrightarrow \textbf{let f in } g \equiv \textbf{let } f' \textbf{ in } g'$$
   (b) if $f : n \rightharpoonup m$ and $g : m \rightharpoonup p$ are elementary rational expressions then
   $$\textbf{let f in } g \equiv (g \circ f)$$
   (c) if $f : n \rightharpoonup m$ and $g : p \rightharpoonup n$ are both rational expressions
      i. (**let** $1_n$ **in f**) $\equiv$ f
      ii. (**let** $g$ **in** $1_n$) $\equiv g$
   (d) if $f : n \rightharpoonup m$, $g : m \rightharpoonup p$ and $h : p \rightharpoonup q$ are rational expressions then
   $$\textbf{let (let f in } g\textbf{) in h} \equiv \textbf{let f in (let } g \textbf{ in h)}$$
3. rules that concern the structure of finite product
   (a) it is a congruence for the pairing operation
   $$f \equiv f' \text{ and } g \equiv g' \Longrightarrow (f;g) \equiv (f';g')$$
   (b) if $f : p \rightharpoonup n$, $g : p \rightharpoonup m$ and $h : p \rightharpoonup n+m$ are rational expressions
      i. **let** $(f,g)$ **in**$\pi_1^{n,m} \equiv f$,
      ii. **let** $(f,g)$ **in**$\pi_2^{n,m} \equiv f$,
      iii. (**let** $h$ **in** $\pi_1^{n,m}$; **let** $h$ **in** $\pi_2^{n,m}$) $\equiv h$,
   (c) if $f : n \rightharpoonup n'$ and $g : m \rightharpoonup m'$ are rational expressions
   $$(\text{f } \textbf{and } g) \equiv (\textbf{let } \pi_1^{n,m} \textbf{ in } f; \textbf{let } \pi_2^{n,m} \textbf{ in } g)$$
   (d) for every rational expression $f : n \rightharpoonup 0$ one has
   $$f \equiv 0_n$$
   (as a result if $f : n \rightharpoonup m$ is a rational expression then **let f in** $0_m \equiv 0_n$)
4. rules that concern the recursive definition :
   (a) it is a congruence for the operator of recursion
   $$f \equiv g \Longrightarrow \textbf{rec}(f) \equiv \textbf{rec}(g)$$
   (b) if $f : n+p \rightharpoonup n$ is an rational expression then
   $$\textbf{let (rec(f) ; } 1_p\textbf{) in f} \equiv \textbf{rec}(f)$$
   (c) if $f : n+p \rightharpoonup n$ and $h : m \rightharpoonup p$ are rational expressions then
   $$\textbf{rec (let (} 1_n \textbf{ and h) in f)} \equiv \textbf{let h in rec(f)}$$
   (d) if $f : n+m+p \rightharpoonup n$ and $g : n+m+p \rightharpoonup m$ are rational expressions then
   $$\textbf{rec}(f;g) \equiv (f_0;g_0)$$
   $$\begin{aligned} \textit{where} \quad & f_0 = \textbf{let } (g_0 \ ; 1_p) \textbf{ in rec(f)} \\ \textit{and} \quad & g_0 = \textbf{rec (let (rec(f) ; } 1_{m+p}\textbf{) in } g) \end{aligned}$$
   (e) if $f : n+p \rightharpoonup n'$ and $g : n' \rightharpoonup n$ are both rational expressions, then
   $$\textbf{rec(let } (f) \textbf{ in } g) \equiv \textbf{let } (h) \textbf{ in } g$$
   where
   $$h = \textbf{rec(let } (g \textbf{ and } 1_p) \textbf{ in } f)$$
   (f) let $f : n+p \rightharpoonup n$ , $g : n'+p \rightharpoonup n'$ be two rational expressions and $\varphi : n' \rightharpoonup n$ be an elementary rational expression correspong-ing to a surjective mapping $\varphi : n \rightarrow n'$ and verifying
   $$\textbf{let } (\varphi \textbf{ and } 1_p) \textbf{ in f} \equiv \textbf{let } g \textbf{ in } \varphi$$
   then
   $$\textbf{rec}(f) \equiv \textbf{let rec } (g) \textbf{ in } \varphi$$

We observe that two equivalent rational expressions necessarily have the same type, let

$$Law^*(\Sigma)(n,m) = Rat(\Sigma)(n,m)/\equiv$$

denote the set of equivalence classes of rational expressions of type $n \rightharpoonup m$. The rules (2) shows that one may define a composition operation

$$\circ : Law^*(\Sigma)(n,m) \times Law^*(\Sigma)(m,p) \to Law^*(\Sigma)(n,p)$$

making $Law^*(\Sigma)$ a category with $Law(\Sigma)$ as subcategory. The rules (3) shows that $Law^*(\Sigma)$ is a Lawvere's algebraic theory and that the inclusion

$$i_\Sigma : Law(\Sigma) \hookrightarrow Law^*(\Sigma)$$

is a morphism of algebraic theories. Finally, the rules (4) enable one to define a fixed point calculus on $Law^*(\Sigma)$. Now let $(\mathcal{T}, \dagger, \alpha)$ be a model and let $f$ and $g$ be two rational expressions of type $n \rightharpoonup m$, if $f \equiv g$ then $\mathcal{M}^\alpha(f) = \mathcal{M}^\alpha(g)$ and then $\mathcal{M}^\alpha$ is compatible with the equivalence relation and provides a morphism of algebraic theories $(\mathcal{M}^\alpha / \equiv) : Law^*(\Sigma) \to \mathcal{T}$; moreover as readily verified $(\mathcal{M}^\alpha / \equiv)$ is the unique morphism of algebraic theories from $Law^*(\Sigma)$ to $\mathcal{T}$and that extends $\alpha : Law(\Sigma) \to \mathcal{T}$ ; i.e. such that $(\mathcal{M}^\alpha / \equiv) \circ i_\Sigma = \alpha$. Moreover, the identity $\mathcal{M}^\alpha_{n,m}(\mathrm{rec}(f)) = (\mathcal{M}^\alpha_{n,m+n}(f))^\dagger$ for any rational expression $f : m+n \rightharpoonup m$ ensures that $(\mathcal{M}^\alpha / \equiv)$ commutes with the fixed point calculi on $Law^*(\Sigma)$ and $\mathcal{T}$ respectively. to summarize :

**Proposition 20** *$Law^*(\Sigma)$ is the algebraically closed theory freely generated by the signature $\Sigma$, that is to say :*

1. *$Law^*(\Sigma)$ is an algebraically closed theory and,*
2. *there exists a morphism of algebraic theories $i_\Sigma : Law(\Sigma) \to Law^*(\Sigma)$ such that for any algebraically closed theory $\mathcal{T}$ and morphism of algebraic theories $\alpha$ from $Law(\Sigma)$ to $\mathcal{T}$ there exits a unique morphism of algebraically closed theories (i.e. that commutes with the fixed point calculi) namely $(\mathcal{M}^\alpha / \equiv)$ such that $(\mathcal{M}^\alpha / \equiv) \circ i_\Sigma = \alpha$.*

In the preceding lines we have described a *'syntactic'* construction of the algebraically closed theory freely generated by a signature $\Sigma$ ; the constraints we required for a fixed point calculus are not really involved in that construction as different conditions would have lead to an analogous universal construction. So nothing is achieve until the algebraic theory so obtained has been characterized, that's what we intend to do in the remaining lines. More precisely, we prove that the preceding axiomatization characterizes the equality of two rational trees defined by rational expressions. Following [ADJ76], we may define the set $PT^\infty(\Sigma, n)$ of partial n-ary trees corresponding to a signature $\Sigma$ as the set of partial mappings $t$ from $\omega^*$ to $\Sigma + n$ such that whenever $t(\alpha \bullet k) = f \in \Sigma$ where $\alpha \in \omega^*$ and $k \in \omega$ there exists an operator symbol $g \in \Sigma_j$ whose arity is at least $k$ (i.e. $k \in j$) and such that $t(\alpha) = g$. They are trees built from $\Sigma$ and in which some branches may be *'hanging'*. We obtain an algebraic theory $\mathbf{PT}^\infty_\Sigma$ whose arrows $\mathbf{PT}^\infty_\Sigma(n,m)$ from $n$ to $m$ constitute the set $PT^\infty(\Sigma, n)^m$ of m-uples of partial n-ary trees and in which the composition is defined as a Kleisli product in the same way as we did in the case of finite trees. We so obtain a rational theory (cf [ADJ76]), actually an $\omega$-continuous theory, and then in particular an algebraically closed theory. Now we let the algebraic theory $\mathbf{PR}_\Sigma$ of rational trees be defined as the **rational closure** [ADJ76] of $Law(\Sigma)$ (i.e. of the terms) in $\mathbf{PT}^\infty_\Sigma$ ; they are trees corresponding to rational expressions. In order to establish the following result (which states the initiality of the Herbrand model $\mathbf{PR}_\Sigma$)

**Proposition 21** $Law^*(\Sigma) \approx \mathbf{PR}_\Sigma$

it remains to verify that if $f$ and $g$ are two rational expressions of type $n \rightharpoonup m$ that define the same m-uple of n-ary rational trees then they are equivalent ($f \equiv g$). The proof is lengthy and is given in the appendix.

# 6 Quotients of algebraically closed theories

We first define the concept of a congruence for an algebraically closed theory.

**Definition 22** *A congruence $\approx$ for an algebraic theory $\mathcal{T}$ is a family of equivalence relations $\approx_{n,m}$ on $\mathcal{T}(n,m)$ which respect composition as well as the structure of finite product of $\mathcal{T}$ ; i.e.*

1. *if $f \approx_{n,p} g$ and $f' \approx_{p,q} g'$ then $f' \circ f \approx_{n,q} g' \circ g$,*
2. *if $f_i \approx_{p,1} g_i$ for $i \in n$ then $[f_0, \ldots, f_{n-1}] \approx_{p,n} [g_0, \ldots, g_{n-1}]$.*

*A congruence $\approx$ for an algebraically closed theory $(\mathcal{T}, \dagger)$ is a congruence for the underlying algebraic theory that moreover respect the fixed point calculus ; i.e.*

$$if \quad f \approx g \quad then \quad f^\dagger \approx g^\dagger$$

If $\alpha : \mathcal{T} \to \mathcal{T}'$ is a morphism of algebraically closed theories, then its kernel $\approx_\alpha$ is a congruence on $(\mathcal{T}, \dagger)$ where

$$f \approx_\alpha g \quad iff \quad \alpha(f) = \alpha(g)$$

Conversely, let $\approx$ be a congruence for an algebraically closed theory $(\mathcal{T}, \dagger)$. Since $\approx$ is a congruence for the algebraic theory $\mathcal{T}$, we obtain a quotient algebraic theory $\mathcal{T}_{/\approx}$ whose arrows are the equivalence classes of arrows of $\mathcal{T}$ for $\approx$. Because $\approx$ respects the fixed point calculus $\dagger$ we can define a family $\dagger_{/\approx}$ of operations

$$(\dagger_{/\approx})_{p,n} : \mathcal{T}_{/\approx}(n+p,n) \to \mathcal{T}_{/\approx}(p,n)$$

by $(\overline{f})^{\dagger_{/\approx}} = \overline{(f^\dagger)}$ where $\overline{(.)} : \mathcal{T} \to \mathcal{T}_{/\approx}$ is the canonical projection. And we prove the following result

**Proposition 23** *The factor of a fixed point calculus $\dagger$ on $\mathcal{T}$ by a congruence $\approx$ for the algebraically closed theory$(\mathcal{T},\dagger)$ is a fixed point calculus for the quotient algebraic theory $\mathcal{T}_{/\approx}$.*

*Proof :*
We give, as an example, the verification for the last condition. Let $\alpha : n+p \to n$, $\beta : n' + p \to n'$ be two arrows in $\mathcal{T}_{/\approx}$ and $\varphi : n' \to n$ be a distinguished morphism corresponding to a surjective mapping and for which

$$\alpha \circ (\varphi \times 1_p) = \varphi \circ \beta$$

There exists a distinguished morphism $\psi$ such that $\psi \circ \varphi = 1_{n'}$. Let $f : n+p \to n$ such that $\overline{(f)} = \alpha$ and let $g = \psi \circ f \circ (\varphi \times 1_p)$. On one hand $\varphi \circ g = f \circ (\varphi \times 1_p)$ hence $f^\dagger = \varphi \circ g^\dagger$, on the other hand $\overline{g} = \beta$. So

$$\alpha^{\dagger_{/\approx}} = \overline{f^\dagger} = \varphi \circ \overline{g^\dagger} = \varphi \circ \beta^{\dagger_{/\approx}}$$

The verifications for the other conditions are analogous. □.

As readily verified the equivalence defined in the introduction turn out to be a congruence (least fixed points are considered in the operationnal model) and the corresponding quotient provides us with an example of model morphism that maps an initial fixed point to a non-initial fixed point.

# 7 Conclusion

A relative interpretation of an algebraic theory into another is a finite product preserving functor between the two underlying categories, when the target theory is the cartesian theory associated to a domain D we recover the usual notion of interpretation. Concerning algebraically closed theories we moreover require from relative interpretations to commute with the respective fixed point calculi. An interesting question is to know whether those semantic functors are models according to Makkai and Reyes [MR77] i.e. continuous functors for some Grothendieck topologies. A positive answer would provide a generalization of M. Eytan's observation ([Eyt84]) that ADJ semantics can be construe as continuous functors.

Notice that all definitions and results stated in this paper extend immediately to the many sorted case using Bénabou' s notion of I-type [Ben68]. We recall that objects in many sorted algebraic theories are words $m \in I^*$ over some set $I$ of '*sorts*' generalizing integers for the ordinary case (which are words over a one element alphabet).

# 8 Appendix : the Herbrand model initiality

In this section we give a detailed proof for the initiality of the Herbrand model, i.e.

$$Law^*(\Sigma) \approx \mathbf{PR}_\Sigma$$

It remains to verify that if $f$ and $g$ are two rational expressions of type $n \to m$ that define the same m-uple of n-ary rational trees then they are equivalent ($f \equiv g$). We may restrict ourself to the case where $m = 1$. The proof is inspired by [CKV74] and consists, in a first stage, in proving that any rational expression $f : n \dot{\to} 1$ is equivalent to an expression having the form **let rec** $d$ **in** $\pi_0^k$ where $k > 0$ and $d : k+n \to k$ is an **elementary, uniform and connex** rational expression. We recall that elementary means that it is an element of $Law(\Sigma)(k+n,k)$ i.e. a k-uple of elements of $T(\Sigma, k+n)$, such a rational expression is said to be uniform if for every $i \in k$ the term $d_i$ either is a variable or contain only one operator symbol and it is connex when every variable in k is useful that is to say accessible from the variable 0. Such expressions have a natural representation as graphs and the fact that they represent the same rational terms is bound to the existence of a rooted bisimulation between their graphs. The last stage then consists in verifying that two graphs with a rooted bisimulation between them correspond to equivalent rational expressions.

## 8.1 Toward a normal form for rational expressions

In order to avoid cumbersome notations we shall identify a rational expression with its corresponding equivalence class in $Law^*$ ; due to this abuse of notation we shall call **elementary** rational expression any rational expression in which the operator symbol **rec** doesn't occur and identify this expression with the corresponding vector of terms. Nevertheless we shall often write $f \equiv g$ to emphasize on the fact that the equality (in *Law**) arises from the axiomatization given above for $\equiv$.

**Lemma 24** *For every rational expression $f : p \rightharpoonup n$ there exists an elementary rational expression $d : k + p \rightharpoonup k$ and a distinguished morphism $t : k \rightharpoonup n$ such that*

$$f \equiv t \circ d^\dagger$$

*Proof :*
The proof is by induction on the structure of $f$.

1. If $f : p \rightharpoonup n$ is an elementary rational expression then
$$f = 1_n \circ (f \circ (0_n \times 1_p))^\dagger$$
Actually $g = (f \circ (0_n \times 1_p))^\dagger \equiv f \circ (0_n \times 1_p) \circ [g, 1_p] \equiv f$.

2. If $f : p \rightharpoonup n_1$ and $g : p \rightharpoonup n_2$ are rational expressions such that $f \equiv t_1 \circ (d_1)^\dagger$ with $d_1 : k_1 + p \rightharpoonup k_1$ and $t_1 : k_1 \rightharpoonup n_1$ and $g \equiv t_2 \circ (d_2)^\dagger$ with $d_2 : k_2 + p \rightharpoonup k_2$ and $t_2 : k_2 \rightharpoonup n_2$ then
$$[f; g] \equiv t \circ d^\dagger : p \rightharpoonup n_1 + n_2$$
with $d = [d_1 \circ (1_{k_1} \times 0_{k_2} \times 1_p); d_2 \circ (0_{k_1} \times 1_{k_2} \times 1_p)] : k_1 + k_2 + p \rightharpoonup k_1 + k_2$ and $t = t_1 \times t_2 : k_1 + k_2 \rightharpoonup n_1 + n_2$. So $d$ is elementary when $d_1$ and $d_2$ both are and $t$ is a distinguished morphism when $t_1$ and $t_2$ both are.

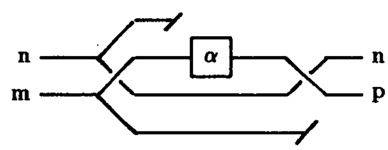
3. If $f : p \rightharpoonup n$ and $g : n \rightharpoonup m$ are rational expressions such that $f \equiv t_1 \circ (d_1)^\dagger$ with $d_1 : k_1 + p \rightharpoonup k_1$ and $t_1 : k_1 \rightharpoonup n$ and $g \equiv t_2 \circ (d_2)^\dagger$ with $d_2 : k_2 + n \rightharpoonup k_2$ and $t_2 : k_2 \rightharpoonup m$ then
$$g \circ f \equiv t \circ d^\dagger : p \rightharpoonup m$$
where $d = [F; G] : k_1 + k_2 + p \rightharpoonup k_1 + k_2$ is given by
$$\begin{aligned} F &= d_1 \circ (1_{k_1} \times 0_{k_2} \times 1_p) \\ G &= d_2 \circ (1_{k_2} \times t_1) \circ (\mathbf{swap}(k_1, k_2) \times 0_p) \end{aligned}$$
in which $\mathbf{swap}(k_1, k_2) = [\pi_2^{k_1,k_2}, \pi_1^{k_1,k_2}]$
and $t = 0_{k_1} \times t_2 : k_1 + k_2 \rightharpoonup m$. So that $d$ is elementary when $d_1$ and $d_2$ both are and $t$ is a distinguished morphism when $t_2$ is.

4. Concerning the product, it proceeds from the preceding cases thanks to the following identity
$$f \times g \equiv [f \circ \pi_1^{n,m}; g \circ \pi_2^{n,m}]$$

5. If $f : n + p \rightharpoonup n$ is a rational expression such that $f \equiv t \circ d^\dagger$ where $d : k + n + p \rightharpoonup k$ is an elementary rational expression and $t : k \rightharpoonup n$ is a distinguished morphism then
$$f^\dagger \equiv T \circ D^\dagger$$
with $D = [d; t \circ (1_k \times 0_{n+p})] : k + n + p \rightharpoonup k + n$ and $T = 0_k \times 1_n : k + n \rightharpoonup n$

The proofs for (2), (3) and (5) are straightforward applications of the Scott-Bekič condition and are left to the reader. For (3), which is a little bit more involved because of the swap operation, we use the following identity

$$(\mathbf{swap}(p, n) \times 0_m) \circ [\alpha \circ (0_n \times 1_m); 1_{n+m}] = 1_n \times \alpha$$




which holds in any algebraic theory.
□.

**Lemma 25** *For every rational expression $f : p \rightharpoonup 1$ there exists an elementary rational expression $d : k + p \rightharpoonup k$ for which $f \equiv \pi_0^k \circ d^\dagger$.*

*Proof :*
$f \equiv \pi_i^k \circ e^\dagger$ for some elementary rational expression $e : k + p \rightharpoonup k$ and index $i \in k$. Now, if $\varphi : k \rightharpoonup k$ is the transposition

$$\varphi(0) = i \quad \varphi(i) = 0 \quad \varphi(j) = j \quad \text{for} \quad j \in k \setminus \{0, i\}$$

then $f \equiv \pi_0^k \circ \varphi \circ e^\dagger \equiv \pi_0^k \circ (\varphi \circ e \circ (\varphi \times 1_p))^\dagger$ □.

**Definition 26** *An elementary rational expression $f : n \rightharpoonup m$ is said to be* **uniform** *if each of its components $f_i$ is a term whose degree is at most égal to one i.e. is either a variable or an element having the form $f[i_0, \ldots, i_{n-1}]$ for some $f \in \Sigma_n$ and variables $i_j$.*

**Lemma 27** *For every rational expression $f : p \rightharpoonup 1$ there exists a uniform rational expression $d : k + p \rightharpoonup k$ for which $f \equiv \pi_0^k \circ d^\dagger$.*

*Proof :*
According to the preceding lemma we can find an elementary rational expression $e : n + p \rightharpoonup n$ for which $f \equiv \pi_0^n \circ e^\dagger$. We add sufficiently many variables to turn $e$ into a uniform rational expression $d : k + p \rightharpoonup k$ for which we have $\pi_0^k \circ d^\dagger \equiv \pi_0^n \circ e^\dagger$. More precisely, we decompose $e$ as $e = \psi \circ \varphi$ such that $\psi$

is the *'upper section'* of e i.e. $\psi$ is a uniform rational expression and $\psi_i$ is a variable if, and only if, $e_i$ was already a variable. This condition will ensure that, except if $e$ already was a distinguished morphism, the maximal depth of a component of $\varphi$ is the maximal depth of a component of $e$ minus 1. For example if

$$e : 3 \rightharpoonup 3 = \begin{cases} x_0 &=& \alpha(\beta(x_1), \alpha(x_2, \delta)) \\ x_1 &=& \gamma(x_2) \\ x_2 &=& \alpha(x_0, \gamma(x_2)) \end{cases}$$

$$\psi : 4 \rightharpoonup 3 = \begin{cases} x_0 &=& \alpha(X_0, X_1) \\ x_1 &=& \gamma(X_2) \\ x_2 &=& \alpha(X_3, X_4) \end{cases}$$

$$\varphi : 3 \rightharpoonup 4 = \begin{cases} X_0 &=& \beta(x_1) \\ X_1 &=& \alpha(x_2, \delta) \\ X_2 &=& x_2 \\ X_3 &=& x_0 \\ X_4 &=& \gamma(x_2) \end{cases}$$

We group those two systems $\varphi$ and $\psi$ together :

$$\psi * \varphi : 7 \rightharpoonup 7 = \begin{cases} x_0 &=& \alpha(x_3, x_4) \\ x_1 &=& \gamma(x_5) \\ x_2 &=& \alpha(x_6, x_7) \\ x_3 &=& \beta(x_1) \\ x_4 &=& \alpha(x_2, \delta) \\ x_5 &=& x_2 \\ x_6 &=& x_0 \\ x_7 &=& \gamma(x_2) \end{cases}$$

More generally, we saw that

$$(\psi \circ \varphi)^\dagger \equiv \pi_1^{n,n'} \circ [\Psi, \Phi]^\dagger$$

with

$$\begin{array}{lcl} \Psi = \psi \circ (0_n \times 1_{n'} \times 0_p) & : & n + n' + p \rightharpoonup n \\ \Phi = \varphi \circ (1_n \times 0_{n'} \times 1_p) & : & n + n' + p \rightharpoonup p \end{array}$$

and in particular $\pi_0^n \circ e^\dagger \equiv \pi_0^{n+n'} \circ (e')^\dagger$ where $e' = [\Psi, \Phi]$. As soon as $e$ is not already a uniform rational expression,the maximal depth of a component of $e'$ is stricly less than the maximal depth of a component of $e$ we then obtain the desired $d$ through a finite iteration of this process .

□.

Now, we can supress in $d$ all *'unguarded'* equations $x_i = x_j$ with $i \neq j$ thanks to the identity

$$(\varphi \circ d)^\dagger \equiv \varphi \circ (d \circ (\varphi \times 1_p))^\dagger$$

actually we let $\varphi : n-1 \rightharpoonup n$ be a mapping (from $n$ to $n-1$) which identifies $i$ and $j$ :

$$\varphi(i) = \varphi(j) = k \in n$$

and set up a bijective correspondance between $n \setminus \{i, j\}$ and $(n-1) \setminus \{k\}$, we can assume that $\varphi(0) = 0$ and then if $e : (n-1) + p \rightharpoonup (n-1)$ is, up to the renumbering of the variables induced by $\varphi$, the system obtained from $d$ by removing the unguarded equation $x_i = x_j$ i.e. $d = \varphi \circ e$ we have

$$f = \pi_0^n \circ (\varphi \circ e)^\dagger = \pi_0^{n-1} \circ (e \circ (\varphi \times 1_p))^\dagger$$

$d' = e \circ (\varphi \times 1_p)$ is obtained from $d$ by removing the unguarded equation $x_i = x_j$ and replacing each occurrence of $x_i$ in the right-hand side of a remaining equation by $x_j$.

Hence the only unguarded equations in $d$ can be assume of the form $x_i = x_i$, such a variable $i$ is said to be a *loop variable* for $d$.

Now we can get rid of unguarded equations by introducing a new 0-ary operator symbol $\perp$. So we let

$$\Sigma_\perp = \Sigma + \{\perp\}$$

and then partial p-ary trees are p-ary trees corresponding to that extended signature :

$$PT^\infty(\Sigma, p) \cong T^\infty(\Sigma_\perp, p)$$

Where the definition of infinite trees $T^\infty(\Sigma, X)$ corresponding to a signature $\Sigma$ and a set of variables $X$ follows for example [Cou83].

**Definition 28** *$T^\infty(\Sigma, X)$ denotes the set of all partial mappings $t : \omega^* \longrightarrow \Sigma + X$ satisfying :*

1. *the domain $Dom(t)$ of $t$ is non empty and is prefix-closed i.e. :*
   *if $\alpha, \beta \in \omega^*$ are such that $\alpha \bullet \beta \in Dom(t)$ then $\alpha \in Dom(t)$*
2. *for $\alpha \in \omega^*$ ; $i, j \in \omega$ if $1 \leq i \leq j$ and $\alpha \bullet j \in Dom(t)$ then $\alpha \bullet i \in Dom(t)$*
3. *if $t(\alpha) = f \in \Sigma_n$ then for $i \in \omega$ one has $\alpha \bullet i \in Dom(t)$ if, and only if $i \in n$*
4. *if $t(\alpha) = x \in X$ then for any $i \in \omega$ $\alpha \bullet i \notin Dom(t)$*

To any uniform rational expression $d : n + p \rightharpoonup n$ we associate the uniform rational expression $d_\perp : n + p \rightharpoonup n$ corresponding to the extended signature $\Sigma_\perp$ such that $(d_\perp)_i = \perp$ if $i$ is a loop variable for $d$ and $(d_\perp)_i = d_i$ otherwise, i.e. in $d$ we replace the unguarded equation $x_i = x_i$ by the equation $x_i = \perp$. $d_\perp$ is an ideal mapping in $Law(\Sigma_\perp)$ considered as a subtheory of $\mathbf{T}^\infty_{\Sigma_\perp}$, this algebraic theory is an iterative theory and then $d_\perp$ admits a unique solution in it. And

**Lemma 29** *The least solution of a uniform rational expression $d : n + p \rightharpoonup n$ in $PT^\infty(\Sigma, p) \cong T^\infty(\Sigma_\perp, p)$ coincides with the unique solution of $d_\perp$ in $T^\infty(\Sigma_\perp, p)$.*

*Proof :*
The least solution for $d$ is the vector of infinite trees

$$\|d\| = \coprod_{n\in\omega} d^{(n)}$$
$$\text{where } \begin{cases} d^{(0)} &= \perp_{n,p} \\ d^{(n+1)} &= d \circ [d^{(n)}, 1_p] \end{cases}$$

if $i$ is a loop variable for $d$ then $\|d\|_i = \perp = (d_\perp)_i$ hence

$$\|d\| = d_\perp \circ [\|d\|, 1_p]$$

□.

## 8.2 Graphical representation of uniform rational expressions

As we saw in introductory examples such uniform rational expressions have a natural representation as graphs. Moreover the rational term denoted by a rational expression is obtained by unfolding the corresponding graph, we shall see that two rational expressions represent the same rational term when a rooted bisimulation may be found between their graphs and in that case they can be proved equivalent. Most of the proofs that appear in this section are standart results concerning bisimulations on graphs and are given here only for the sake of completeness.
The set $G(\Sigma, X)$ of graphs corresponding to a signature $\Sigma$ and a set of variables $X$ is made of some quadruples $g = <\mathcal{N}, \mathcal{E}, \lambda, n_0>$ for which

- $\mathcal{N}$ is a finite set of *'nodes'*,
- $\mathcal{E}$ is a subset of $\mathcal{N} \times \omega \times \mathcal{N}$ , an element $<n, i, m> \in \mathcal{E}$ is an *edge* from the node $n$ to the node $m$ labelled by the integer $i$,
- $\lambda : \mathcal{N} \longrightarrow \Sigma + X$ is a mapping which takes each node either to an operator symbol in $\Sigma$ or to a variable in $X$, and
- $n_0$ is a distinguished node called the *root* of $g$.

Let us fix some notations. For every node $n$, we let

$$\delta(n) = \{<i, m> \ / \ <n, i, m> \in A\}$$

and we define a transition relation $\mathcal{E}^* \subset \mathcal{N} \times \omega^* \times \mathcal{N}$ corresponding to $g$ by :

$$\begin{array}{lcl} n \xrightarrow{\epsilon} n & & \\ n \xrightarrow{i} m & \Longleftrightarrow & <n, i, m> \in \mathcal{E} \\ n \xrightarrow{i\bullet\alpha} m & \Longleftrightarrow & \exists n' \in \mathcal{N} \ . \ n \xrightarrow{i} n' \ et \ n' \xrightarrow{\alpha} m \end{array}$$

where $n$ and $m$ are nodes, $i \in \omega$, $\alpha \in \omega^*$ and the expression $n \xrightarrow{\alpha} m$ stands for $<n, \alpha, m> \in \mathcal{E}^*$. Lastly, to each node $n$ is associated its ***language of accessibility***

$$\mathcal{L}(n) = \{\alpha \in \omega^* \ ; \ n_0 \xrightarrow{\alpha} n\}$$

Such a 4-uple $g = <\mathcal{N}, \mathcal{E}, \lambda, n_0>$ is an element of $G(\Sigma, X)$ whenever the two following conditions are fulfilled

1. if $n \in \mathcal{N}$ is a node whose label is an n-ary operator symbol, then it must have precisely n successors, that is to say : if $\lambda(n) = f \in \Sigma_n$ then
$$\exists n_0, \ldots, n_{n-1} \in \mathcal{N} \ . \ \delta(n) = \{<i, n_i> \ ; \ i \in n\}$$
2. a variable node has no successor :
$$\forall n \in \mathcal{N} \quad (\lambda(n) = x \in X \implies \delta(n) = \emptyset)$$

Two graphs $g_1 = <\mathcal{N}_1, \mathcal{E}_1, \lambda_1, n_{0,1}>$ and $g_2 = <\mathcal{N}_2, \mathcal{E}_2, \lambda_2, n_{0,2}>$ which differ only from their sets of nodes, in that there exists a bijective mapping $\varphi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ verifying $n_{0,2} = \varphi(n_{0,1})$ , $\lambda_2(\varphi(n)) = \lambda_1(n)$ and $\mathcal{E}_2 = \{<\varphi(n), i, \varphi(m)> \ / \ <n, i, m> \in \mathcal{E}_1\}$ are said to be **isomorphic**.
To each uniform rational expression $f : n+p \rightarrow n$ we associate a graph in $G(\Sigma_\perp, p)$ where $\Sigma_\perp = \Sigma + \{\perp\}$ is the signature obtained from $\Sigma$ by adding a new 0-ary operator symbol $\perp$. This graph **graph**$(f)$ is defined through the following statements

- its set of nodes is $\mathcal{N} = n + p$ with initial node $n_0 = 0$,
- if $f(i) = \alpha[i_0, \ldots, i_{n-1}]$ then $\delta(i) = \{<j, i_j>; j \in n\}$ and $\lambda(i) = \alpha \in \Sigma$,
- if $i$ is a loop variable we let $\delta(i) = \emptyset$ and $\lambda(i) = \perp$ and
- if $f(i) = j \in p$ then $\delta(i) = \emptyset$ and $\lambda(i) = j \in p$.

**Definition 30** *A uniform rational expression $f : n + p \rightarrow n$ is said to be* **connex** *if all inner nodes (nodes not corresponding to a parameter) of* **graph**$(f)$ *are accessible from the initial node, i.e.*

$$\forall n \in \mathcal{N} \quad \mathcal{L}(n) \neq \emptyset$$

**Lemma 31** *For every uniform rational expression $d : n + p \rightarrow n$ there exists a connex and uniform rational expression $d' : n' + p \rightarrow n'$ for which*

$$\pi_0^{n'} \circ (d')^\dagger \equiv \pi_0^n \circ d^\dagger$$

*Proof :*
Thanks to the relation

$$(g \circ d)^\dagger \equiv g \circ (d \circ (g \times 1_p))^\dagger$$

for bijective $g$, we know that the order in which the equations are displayed has no effect on the computation of the solution ; we can therefore assume that $n = n' + n''$ such that the variables in $n'$ are accessibles from 0 and those in $n''$ aren't. That is to say $d \equiv [D', D'']$ with

$$\begin{array}{rcl} D' & = & d' \circ (1_{n'} \circ 0_{n''} \times 1_p) : n+p \longrightarrow n' \\ D'' & = & d'' \circ (0_{n'} \circ 1_{n''} \times 1_p) : n+p \longrightarrow n'' \end{array}$$

for some $d' : n' + p \longrightarrow n'$ and $d'' : n'' + p \longrightarrow n''$. Using the Scott-Bekič condition we readily verify that

$$\pi_1^{n',n''} \circ d^\dagger \equiv (d')^\dagger$$

□.

If $f : n+p \longrightarrow n$ is a connex and uniform rational expression its corresponding rational tree $f^* = \pi_0^n \circ \|f\|^\dagger$ is the unfolding of **graph**$(f)$ where

$$\mathbf{unfold} : G(\Sigma_\perp, X) \longrightarrow R(\Sigma_\perp, X)$$

is the mapping that takes each graph $g = <\mathcal{N}, \mathcal{E}, \lambda, n_0>$ to the rational tree $t$ such that

- $Dom(t) = \bigcup_{n \in \mathcal{N}} \mathcal{L}(n)$,
- $\forall \alpha \in \mathcal{L}(n) \quad t(\alpha) = \lambda(n)$.

(we observe that the sets $\mathcal{L}(n)$ are pairwise disjoint so the previous definition makes sense).
Conversely we can define a folding operation as the mapping **fold** : $R(\Sigma_\perp, X) \longrightarrow G(\Sigma_\perp, X)$ which takes the rational tree $t$ to the graph $g = <\mathcal{N}, \mathcal{E}, \lambda, n_0> =$ **fold**$(t)$ defined as follow. We first introduce the following notation : if $\alpha \in Dom(t)$ then $t\backslash\alpha$ stands for the subtree of $t$ defined by

$$Dom(t\backslash\alpha) = \{\beta \in \omega^* \ / \ \alpha \bullet \beta \in Dom(t)\}$$

and

$$(t\backslash\alpha)(\beta) = t(\alpha \bullet \beta)$$

Now, since $t$ is rational the equivalence relation

$$\alpha \cong \beta \Longleftrightarrow t\backslash\alpha = t\backslash\beta$$

has only a finite number of classes, we let

$$[\alpha] = \{\beta \in Dom(t) \ ; \ t\backslash\alpha = t\backslash\beta\}$$

be the class of $\alpha$, $g$ is then given by

- $\mathcal{N} = \{[\alpha] \ / \ \alpha \in Dom(t)\}$,
- $<n, i, m> \in \mathcal{E} \Longleftrightarrow \exists \alpha, \beta \in Dom(t)$ *s.t.* $\alpha \in n$ , $\beta \in m$ *and* $\beta = \alpha \bullet i$,
- if $\alpha \in n$ then $\lambda(n) = t(\alpha)$ and
- $n_0 = [\epsilon] = \{\alpha \in Dom(t) \ ; \ t = t\backslash\alpha\}$.

If $t$ is a rational tree, $u$ one of its subtree and $n =$ **Occ**$(u, t) = \{\alpha \in Dom(t) \ ; \ t\backslash\alpha = u\}$ the set of its occurrences in $t$, we verify that $n$ coincides with its accessibility language in $g =$ **fold**$(t)$ that is to say

$$n = \{\alpha \in \omega^* \ ; \ n_0 \xrightarrow{\alpha} n\}$$

and consequently, **unfold** ∘ **fold** $= 1_{R(\Sigma_\perp, X)}$ ; it follows that the composite **min** = **fold** ∘ **unfold** is an an idempotente operation on $G(\Sigma_\perp, X)$ ; it associates each graph to its '*minimal*' graph.
We define a notion of reduction on graphs

**Definition 32** *Let* $f = <\mathcal{N}_f, \mathcal{E}_f, n_0>$ *and* $g = <\mathcal{N}_g, \mathcal{E}_g, n_1>$ *be two graphs in* $R(\Sigma_\perp, X)$ *and* $\varphi : \mathcal{N}_f \longrightarrow \mathcal{N}_g$ *a surjective mapping between their sets of nodes.* $g$ *is said to be a* $\varphi$*-reduction of* $f$*, which we denote as* $f \xrightarrow{\varphi} g$*, when the following conditions are fulfilled*

1. $\varphi$ *respects the roots :* $\varphi(n_0) = n_1$
2. $\varphi$ *respects the labels :* $\forall n \in \mathcal{N}_f$ $\lambda_f(n) = \lambda_g(\varphi(n))$
3. $\varphi$ *respects the successors :* $\forall n \in \mathcal{N}_f$ $\delta_g((\varphi(n)) = \{<i, \varphi(j)> \ ; \ <i, j> \in \delta_f(n)\}$

*we write* $f \xrightarrow{\sim} g$ *when such a surjective mapping* $\varphi$ *exists.*

Since graphs have only a finite number of nodes

$$f \xrightarrow{\sim} g \quad \text{and} \quad g \xrightarrow{\sim} f \implies f \approx g$$

and up to isomorphism of graphs, $\xrightarrow{\sim}$ is clearly a noetherian order relation. And we check to two following facts.

**Lemma 33** *If* $f \xrightarrow{\sim} g$ *then* **unfold**$(f) =$ **unfold**$(g)$

*Proof :* It comes from the fact that if $f \xrightarrow{\sim} g$ then

$$\forall n' \in \mathcal{N}_g \quad \mathcal{L}_g(n') = \bigcup_{n \in \varphi^{-1}(n')} \mathcal{L}_f(n)$$

and $\lambda_g(n') = \lambda_f(n)$ whenever $n \in \varphi^{-1}(n')$.
□
The operation **min** takes each graph to its minimal graph regarding this reduction operation on graphs, that is to say :

**Lemma 34** *For all graphs* $g$ *and* $h$ *in* $G(\Sigma, X)$ *one has*

*1.* $g \xrightarrow{\sim} \mathbf{min}(g)$ *and,*

*2.* $\mathbf{min}(g) \xrightarrow{\sim} h \implies \mathbf{min}(g) \approx h$.

*Proof :*

1. For each graph $g = <\mathcal{N}, \mathcal{E}, n_0> \in G(\Sigma, X)$ and $\alpha \in \mathcal{L}(n_1)$ where $n_1 \in \mathcal{N}$ we let

$$g\backslash\alpha = <\mathcal{N}\backslash\alpha, \mathcal{E}\backslash\alpha, n_1>$$

be the graph given by

(a) $\mathcal{N}\backslash\alpha = \{n \in \mathcal{N} \ ; \ \exists\beta = \alpha \bullet \gamma \in \mathcal{L}(n)\}$ and,

(b) $<n,i,m> \in \mathcal{E}\backslash\alpha \iff <n,i,m> \in \mathcal{E}$.

We immediately see that

$$\alpha, \beta \in \mathcal{L}(n_1) \implies g\backslash\alpha = g\backslash\beta$$

which allows us to use the notation $g\backslash n_1$. Moreover, if $\alpha \in n_1$

$$\mathbf{unfold}(g\backslash n_1) = \mathbf{unfold}(g)\backslash\alpha$$

so that we can define a surjective mapping

$$\varphi : \mathcal{N}_g \rightarrow \mathcal{N}_{\mathbf{min}(g)}$$

that takes a node $n \in \mathcal{N}_g$ to the equivalence class $[\alpha]$ for the relation $\cong$ of any element $\alpha \in \mathcal{L}(n)$. $\mathbf{min}$(g) is then clearly a $\varphi$-reduction of $g$.

2. The second part is an easy verification :

$$\begin{aligned}
\mathbf{min}(g) \xrightarrow{\sim} h &\implies \mathbf{unfold}(\mathbf{min}(g)) = \mathbf{unfold}(h)\\
&\implies \mathbf{min}(g) = \mathbf{min}(\mathbf{min}(g)) = \mathbf{min}(h)\\
&\implies h \xrightarrow{\sim} \mathbf{min}(h) = \mathbf{min}(g)
\end{aligned}$$

and then $\mathbf{min}(g) \approx h$ □.

**Definition 35** *A* **rooted bisimulation** *between two graphs $f = <\mathcal{N}_f, \mathcal{E}_f, n_0>$ and $g = <\mathcal{N}_g, \mathcal{E}_g, n_1>$ in $G(\Sigma, X)$ is a relation $\mathcal{R} \subset \mathcal{N}_f \times \mathcal{N}_g$ between their nodes that satisfies the following*

*1. roots are related : $(n_0, n_1) \in \mathcal{R}$ and,*

*2. if $(n, m) \in \mathcal{R}$ then $\lambda_f(n) = \lambda_g(m)$ and*

$$\forall n \xrightarrow{i} n' \in f \ \exists m' \in \mathcal{N}_g \ \text{ s.t. } (n', m') \in \mathcal{R} \text{ and } m \xrightarrow{i} m' \in g$$

$$\forall m \xrightarrow{i} m' \in g \ \exists n' \in \mathcal{N}_f \ \text{ s.t. } (n', m') \in \mathcal{R} \text{ and } n \xrightarrow{i} n' \in f$$

*We write $f \overset{\mathcal{R}}{\leftrightarrow} g$ to indicate that $\mathcal{R}$ is a rooted bisimulation between the two graphs and $f \overset{\sim}{\leftrightarrow} g$ when such a relation exists.*

Concerning rooted bisimulation we prove the two following results.

**Lemma 36** *The rooted bisimulation is the kernel of the reduction operation, i.e.*

$$f \overset{\sim}{\leftrightarrow} g \iff \exists h \,.\, f \overset{\sim}{\rightarrow} h \text{ and } g \overset{\sim}{\rightarrow} h$$

*Proof :*
For one part, if $f \overset{\varphi_1}{\rightarrow} h$ and $g \overset{\varphi_2}{\rightarrow} h$ we verify that $f \overset{\mathcal{R}}{\leftrightarrow} g$ where $\mathcal{R}$ is the push-out of $\varphi_1$ and $\varphi_2$ i.e.

$$\mathcal{R} = \{(i,j) \in \mathcal{N}_f \times \mathcal{N}_g \ ; \ \varphi_1(i) = \varphi_2(j)\}$$

Conversely if $f \overset{\mathcal{R}}{\leftrightarrow} g$ let $(\mathcal{N}_h, \hat{\pi}_1, \hat{\pi}_2)$ be the pullback of $(\mathcal{R}, \pi_1, \pi_2)$ where $\pi_1 : \mathcal{R} \rightarrow \mathcal{N}_f$ and $\pi_2 : \mathcal{R} \rightarrow \mathcal{N}_g$ are the two projections corresponding to $\mathcal{R}$. In detail, $\mathcal{R}$ can be viewed as a binary relation on $\mathcal{N}_f + \mathcal{N}_g$, we let $\equiv$ be $(\mathcal{R} \cup \mathcal{R}^{-1})^*$ i.e. the equivalence relation generated by $\mathcal{R}$ ; then $\mathcal{N}_h = (\mathcal{N}_f + \mathcal{N}_g)/\equiv$ and $\hat{\pi}_1 : \mathcal{N}_g \rightarrow \mathcal{N}_h$ and $\hat{\pi}_2 : \mathcal{N}_f \rightarrow \mathcal{N}_h$ are the two components of the canonical projection. Clearly, each of $\hat{\pi}_1$ and $\hat{\pi}_2$ are surjective mappings and there is exactly one manner to define a graph $h$ having $\mathcal{N}_h$ as set of nodes and verifying

$$f \overset{\hat{\pi}_2}{\rightarrow} h \quad \text{and} \quad g \overset{\hat{\pi}_1}{\rightarrow} h$$

□

**Lemma 37** *The rooted bisimulation is the kernel of the unfolding operation. i.e.*

$$f \overset{\sim}{\leftrightarrow} g \iff \mathbf{unfold}(f) = \mathbf{unfold}(g)$$

*Proof :*

$$f \overset{\sim}{\leftrightarrow} g \ \Rightarrow \exists h \ f \overset{\sim}{\rightarrow} h \overset{\sim}{\leftarrow} g$$

hence $\mathbf{unfold}(f) = \mathbf{unfold}(h) = \mathbf{unfold}(g)$. Conversely

$$\mathbf{unfold}(f) = \mathbf{unfold}(g) \ \Rightarrow \ \mathbf{min}(f) = \mathbf{min}(g) = h$$

hence $f \overset{\sim}{\rightarrow} h \overset{\sim}{\leftarrow} g$ □.
In order to complete our proof it just remains to verify that two uniform and connex rational expressions whose corresponding graphs are related by a rooted bisimulation are equivalent. In view of the preceding result, is suffices to check that both notions of reduction introduced for rational expressions and graphs are related as follow .

**Lemma 38** *If $f : n + p \rightarrow p$ and $g : m + p \rightarrow p$ are uniform and connex rational expressions then*

$$f \xrightarrow{\varphi} g \quad \textit{iff} \quad \mathbf{graph}(f) \xrightarrow{\varphi} \mathbf{graph}(g)$$

*Proof*: **graph**$(f) \xrightarrow{\varphi}$ **graph**$(g)$ iff

$$\begin{array}{lll} f_i = \alpha[i_0, \ldots, i_{n-1}] & \Rightarrow & g_{\varphi(i)} = \alpha[\varphi(i_0), \ldots, \varphi(i_{n-1})] \\ f_i = j \in p & \Rightarrow & g_{\varphi(i)} = j \in p \end{array}$$

if $i$ is a loop variable for $f$ then $\varphi(i)$ is a loop variable for $g$
i.e. $\forall i \in n \quad (\varphi \circ g)_i = (f \circ (\varphi \times 1_p))_i$
i.e. $\varphi \circ g = f \circ (\varphi \times 1_p)$. □

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